On the solution of the 4th clay millennium problem. Proof of the regularity of the solutions of the Euler and Navier-Stokes equations, based on the conservation of particles as a local structure of the fluid, formulated in the context of continuous fluid mechanics.

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ABSTRACT

As more and more researchers tend to believe that with the hypotheses of the official formulation of the 4th Clay Millennium problem a blowup may occur, a new goal is set: to find the simplest and most physically natural enhancement of the hypotheses in the official formulation so that the regularity can be proved in the case of 3 dimensions too. The position of this paper is that the standard assumptions of the official formulation of the 4th Clay millennium problem, although they reflect, the finiteness and the conservation of momentum and energy and the smoothness of the incompressible physical flows, they do not reflect the conservation of particles as local structure. By formulating the later conservation and adding it to the hypotheses, we prove the regularity (global in time existence and smoothness) both for the Euler and the Navier-Stokes equations.

Key words: *Incompressible flows, regularity, Navier-Stokes equations, 4th Clay millennium problem*

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1. Introduction

The famous problem of the 4th Clay mathematical Institute as formulated in FEFFERMAN C. L. 2006 , is considered a significant challenge to the science of mathematical physics of fluids, not only because it has withstand the efforts of the scientific community for decades to prove it (or types of converses to it) but also because it is supposed to hide a significant missing perception about the nature of our mathematical formulations of the physical flows through the Euler and the Navier-Stokes equations.

When the 4th Clay Millennium problem was officially formulated the majority was hoping that the regularity was holding also in 3 dimensions as it had been proved to hold also in 2 dimensions. But as time passed more and more mathematicians started believing that a Blowup can occur with the hypotheses of the official formulation. Therefore a new goal is set to find the simplest and most physically natural

enhancement of the hypotheses in the official formulation so that the regularity can be proved in the case of 3 dimensions too. This is done by the current paper.
After 3 years of research, in the 4^{th} Clay Millennium problem, the author came to believe that, what most of the mathematicians would want, (and seemingly including the official formulators of the problem too), in other words a proof of the regularity in 3 dimensions as well, cannot be given merely by the assumptions of the official formulation of the problem. In other words a Blow-up may occur even with compact support smooth initial data with finite energy. But solving the 4^{th} Clay Millennium problem, by designing such a case of Blow-up is I think not interesting from the physical point of view, as it is quite away from physical applications and a mathematical pathological curiosity. On the other hand discovering what physical aspect of the flows is not captured by the mathematical hypotheses, is I believe a more significant contribution to the science of mathematical physics in this area. Although the mathematical assumptions of the official formulation reflect, the finiteness and the conservation of momentum and energy and the smoothness of the incompressible physical flows, they do not reflect the conservation of particles as local structure. By adding this physical aspect formulated simply in the context of continuous fluid mechanics, the expected result of regularity can be proved.
In statistical mechanical models of incompressible flow, we have the realistic advantage of finite many particles, e.g. like balls B(x,r) with finite diameter r. These particles as they flow in time, **remain particles of the same nature and size** and the velocities and inside them remain approximately constant.
Because space and time dimensions in classical fluid dynamics goes in orders of smallness , smaller and at least as small as the real physical molecules , atoms and particles of the fluids, this might suggest imposing too, such conditions resembling uniform continuity conditions. In the case of continuous fluid dynamics models such natural conditions, emerging from the particle nature of material fluids, together with the energy conservation, the incompressibility and the momentum conservation, as laws conserved in time, may derive the regularity of the local smooth solutions of the Euler and Navier-Stokes equations. **For every atom or material particle of a material fluid, we may assume around it a ball of fixed radius, called *particle range* depending on the size of the atom or particle, that covers the particle and a little bit of the electromagnetic, gravitational or quantum vacuum field around it, that their velocities and space-time accelerations are affected by the motion of the molecule or particle.** E.g. for the case water, we are speaking here for molecules of H_2O, that are estimated to have a diameter of 2.75 angstroms or 2r= 2.75*10^(-10) meters, we may define as water molecule **particle range** the balls $B(r_0)$ of radius r_0=3*10^(-10) meters around the water molecule. As the fluid flows, especially in our case here of incompressible fluids, the shape and size of the molecules do not change much, neither there are significant differences of the velocities and space-time accelerations of parts of the molecule. Bounds δ_u δ_ω of such differences remain constant as the fluid flows. **We may call this effect as the *principle of conservation of particles* as a local structure.** This principle must be posed in equal setting as the energy conservation and incompressibility together with the Navier-Stokes or Euler equations. Of course if the fluid is say of solar plasma matter, such a description would not apply. Nevertheless then incompressibility is hardly a property of it. But if we are talking about incompressible fluids that the molecule is conserved as well as the atoms and do not change atomic number (as e.g. in fusion or fission) then this principle is physically valid. The principle of conservation of particles as a local structure, blocks the self-similarity effects of concentrating the energy and turbulence

in very small areas and creating thus a Blow-up. It is the missing invariant in the discussion of many researchers about superctitical, critical and subcritical invariants in scale transformations of the solutions.

The exact definition of the conservation of particles as local structure Is in DEFINITION 5.1 and it is as follows:

(**Conservation of particles as local structure in a fluid**)

Let a smooth solution of the Euler or Navier-Stokes equations for incompressible fluids, that exists in the time interval [0,T). We may assume initial data on all of R^3 or only on a connected compact support V_0 . For simplicity let us concentrate only on the latter simpler case. Let us denote by F the displacement transformation of the flow. Let us also denote by g the partial derivatives of 1st order in space and time , that is $\left|\partial_x^a \partial_t^b u\ (x)\right|$, *$|\alpha|=1$, $|b|<=1$, and call then space-time accelerations . We say that there is* ***conservation of the particles*** ***in the interval [0,T) in*** *a derivatives homogenous setting, as a local structure of the solution if and only if:*

There is a small radius r, and small constants δ_x , δ_u , δ_ω , >0 so that for all t in [0,T) there is a finite cover C_t (in the case of initial data on R^3 , it is infinite cover, but finite on any compact subset) of V_t , from balls B(r) of radius r, called ***ranges of the particles*** *, such that:*

1) *For an x_1 and x_2 in a ball B(r) of V_s , s in [0,T), $||F(x_1)-F(x_2)||<=r+\delta_x$ for all t>=s in [0,T).*
2) *For an x_1 and x_2 in a ball B(r) of V_s ,s in [0,T), $||u(F(x_1))-u(F(x_2))||<=\delta_u$ for all t >=s in [0,T).*
3) *For an x_1 and x_2 in a ball B(r) of V_s , s in [0,T), $||g(F(x_1))-g(F(x_2))||<=\delta_\omega$ for all t >=s in [0,T).*

   If we state the same conditions 1) 2) 3) for all times t in $[0,+\infty)$, then we say that we have the ***strong version*** *of the conservation of particles as local structure.*

We prove in paragraph 5 in PROPOSITION 5.2 that indeed adding the above conservation of particles as local structure in the hypotheses of the official formulation of the 4th Clay Millennium problem, we solve it, in the sense of proving the regularity (global in time smoothness) of the locally in time smooth solutions that are known to exist.

A short outline of the logical structure of the paper is the next.

1) The paragraph 3, contains the official formulation of the 4th Cay millennium problem as in FEFFERMAN C. L. 2006. The official formulation is any one of 4 different conjectures, that two of them, assert the existence of blow-up in the periodic and non-periodic case, and two of them the non-existence of blow-up , that is the global in time regularity in the periodic and non-periodic case. We concentrate on to prove the regularity in the non-periodic case or conjecture (A) with is described by equations 1-6 after adding the conservation of particles as a local structure. The paragraph 3 contains definitions, and more modern symbolism introduced by T, Tao in TAO T. 2013. The current paper follows the formal and mathematical austerity standards that the official formulation has set, together with the suggested by the official formulation relevant results in the literature like in the book MAJDA A.J-BERTOZZI A. L. 2002 .

But we try also not to lose the intuition of the physical interpretation, as we are in the area of mathematical physics rather than pure mathematics.
The goal is that reader after reading a dozen of mathematical propositions and their proofs , he must be able at the end to have simple physical intuition , why the conjecture (A) of the 4^{th} Clay millennium together with the conservation of particles in the hypotheses problem holds.

2) The paragraph 4 contains some known theorems and results, that are to be used in this paper, so that the reader is not searching them in the literature and can have a direct, at a glance, image of what holds and what is proved. The most important are a list of necessary and sufficient conditions of regularity (PROPOSITIONS 4.5-4.10) The same paragraph contains also some well known and very relevant results that are not used directly but are there for a better understanding of the physics.
3) The paragraph 5 contains the main idea that the conservation of particles during the flow can be approximately formulated in the context of continuous fluid mechanics and that is the key missing concept of conservation that acts as subcritical invariant in other words blocks the self-similar concentration of energy and turbulence that would create a Blowup. With this new invariant we prove the regularity in the case of 3 dimensions: PROPOSITIONS 5.2 .
4) The paragraph 6 contains the idea of defining **a measure of turbulence** in the context of deterministic mechanics based on the **total variation** of the component functions or norms (DEFINITION 6.1) It is also made the significant observation that the smoothness of the solutions of the Euler and Navier-Stokes equations is not a general type of smoothness but one that would deserve the name **homogeneous smoothness** (Remark 6.2) .

According to CONSTANTIN P. 2007 "..The blowup problem for the Euler equations is a major open problem of PDE, theory of far greater physical importance that the blow-up problem of the Navier-Stokes equation, which is of course known to non specialists because of the Clay Millennium problem…"
Almost all of our proved propositions and in particular the regularity in paragraphs 4 , 5 and 6 (in particular PROPOSITION 4.11 and PROPOSITION 5.2) are stated not only for the Navier-Stokes but also for the Euler equations.

2. The ontology of the continuous fluid mechanics models versus the ontology of statistical mechanics models. The main physical idea of the proof of the regularity in 3 spatial dimensions.

All researchers discriminate between the physical reality with its natural physical ontology (e.g. atoms, fluids etc) from the mathematical ontology (e.g. sets, numbers, vector fields etc). If we do not do that much confusion will arise. The main difference of the physical reality ontology, from the mathematical reality ontology, is what the mathematician D. Hilbert had remarked in his writings about the infinite. He remarked that nowhere in the physical reality there is anything infinite, while the mathematical infinite, as formulated in a special axiom of the infinite in G. Cantor's theory of sets, is simply a convenient phenomenological abstraction, at a time that the atomic theory of matter was not well established yet in the mathematical community. In the physical reality ontology, as best captured by statistical mechanics models, the problem of the global 3-dimensional regularity seems easier to solve. For example it is known (See PROPOSITION 4.9 and PROPOSITION 4.12 maximum Cauchy development, and it is referred also in the official formulation of the Clay millennium problem in C. L. FEFFERMAN 2006) that if the global 3D regularity does not hold then the velocities become unbounded or tend in absolute value to infinite as time gets close to the finite Blow-up time. Now we know that a fluid consists from a finite number of atoms and molecules, which also have finite mass and with a lower bound in their size. **If such a phenomenon (Blowup) would occur, it would mean that for at least one particle the kinetic energy, is increasing in an unbounded way.** But from the assumptions (see paragraph 3) the initial energy is finite, so this could never happen. We conclude that the fluid is 3D globally in time regular. Unfortunately such an argument although valid in statistical mechanics models (see also MURIEL A 2000), in not valid in continuous fluid mechanics models, where there are not atoms or particles with lower bound of finite mass, but only points with zero dimension, and only mass density. We must notice also here that this argument is not likely to be successful if the fluid is compressible. In fact it has been proved that a blow-up may occur even with smooth compact support initial data, in the case of compressible fluids. One of the reasons is that if there is not lower bound in the density of the fluid, then even without violating the momentum and energy conservation, a density converging to zero may lead to velocities of some points converging to infinite.

Nevertheless if we formulate in the context of continuous fluid mechanics the conservation of particles as a local structure (DEFINITION 5.1) then we can derive a similar argument (see proof of PROPOSITION 5.1) where **if a Blowup occurs in finite time then the kinetic energy of a finite small ball (called in** DEFINITION 5.1 **particle-range) will become unbounded**, which is again impossible, due to the hypotheses if finite initial energy and energy conservation.

The next table compares the hypotheses and conclusions both in continuous fluid mechanics models and statistical mechanics models of the 4th Clay millennium problem in its officially formulation together with the hypothesis of conservation of particles. It would be paradoxical that we would be able to prove the regularity in statistical mechanics and we would not be able to prove it in continuous fluid mechanics.

Table 1

COMPARISON AND MUTUAL SIGNIFICANCE OF DIFFERENT TYPES OF MATHEMATICAL MODELS FOR THE 4TH CLAY PROBLEM (NO EXTERNAL FORCE)	**CONTINUOUS FLUID MECHANICS MODEL**	**STATISTICAL MECHANICS MODEL**
SMOOTH SCHWARTZ INITIAL CONDITIONS	YES	POSSIBLE TO IMPOSE
FINITE INITIAL ENERGY	YES	YES
CONSERVATION OF THE PARTICES	***YES(NON-OBVIOUS FORMULATION)***	***YES (OBVIOUS FORMULATION)***
LOCAL SMOOTH EVOLUTION IN A	YES	POSSIBLE TO DERIVE

INITIAL FINITE TIME INTERVAL		
EMERGENCE OF A BLOW-UP IN FINITE TIME	IMPOSSIBLE TO OCCUR	IMPOSSIBLE TO OCCUR

3. The official formulation of the Clay Mathematical Institute 4th Clay millennium conjecture of 3D regularity and some definitions.

In this paragraph we highlight the basic parts of the official formulation of the 4th Clay millennium problem, together with some more modern, since 2006, symbolism, by relevant researchers, like T. Tao.

In this paper I consider the conjecture (A) of C. L. FEFFERMAN 2006 official formulation of the 4th Clay millennium problem , which I indentify throughout the paper as <u>the 4th Clay millennium problem</u>.

The Navier-Stokes equations are given by (by R we denote the field of the real numbers, ν>0 is the viscosity coefficient)

$$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \Delta u_i \qquad (x\varepsilon R^3 , t>=0 , n=3) \qquad \text{(eq.1)}$$

$$divu = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \qquad (x\varepsilon R^3 , t>=0 , n=3) \qquad \text{(eq.2)}$$

with initial conditions $u(x,0)=u^0(x)$ **xεR^3**
and u_0 (x) C∞ divergence-free vector field on $\mathbf{R^3}$ (eq.3)

$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator .The Euler equations are when ν=0

For physically meaningful solutions we want to make sure that $u^0(x)$ does not grow large as $|x|\to\infty$. This is set by defining $u^0(x)$ and called in this paper **Schwartz initial conditions** , in other words

$$\left|\partial_x^a u^0(x)\right| \le C_{a,K}(1+|x|)^{-K} \text{ on } \mathbf{R^3} \text{ for any } \alpha \text{ and K} \qquad \text{(eq.4)}$$

(Schwartz used such functions to define the space of Schwartz distributions)
We accept as physical meaningful solutions only if it satisfies

$$p, u \in C^\infty(R^3 \times [0,\infty)) \qquad \text{(eq.5)}$$

and

$$\int_{\Re^3} |u(x,t)|^2 \, dx < C$$ for all t>=0 (**Bounded or finite energy**) (eq.6)

The conjecture (A) of he Clay Millennium problem (case of no external force, but homogeneous and regular velocities) claims that for the Navier-Stokes equations , v>0, n=3 , with divergence free , Schwartz initial velocities , there are for all times t>0 , smooth velocity field and pressure, that are solutions of the Navier-Stokes equations with bounded energy**, in other words satisfying the equations eq.1 , eq.2 , eq. 3, eq.4 , eq.5 eq.6** . It is stated in the same formal formulation of the Clay millennium problem by C. L. Fefferman see C. L. FEFFERMAN 2006 (see page 2nd line 5 from below) that the conjecture (A) has been proved to holds locally. "..if the time internal $[0,\infty)$, is replaced by a small time interval [0,T), with T depending on the initial data....". In other words there is $\infty>T>0$, such that there is continuous and smooth solution $u(x,t)\in C^{\infty}(R^3 \times[0,T))$. In this paper, as it is standard almost everywhere, the term smooth refers to the space C^{∞}

Following TAO, T 2013, we define some specific terminology , about the hypotheses of the Clay millennium problem, that will be used in the next.
We must notice that the definitions below can apply also to the case of inviscid flows, satisfying the ***Euler*** *equations.*

DEFINITION 3.1 (Smooth solutions to the Navier-Stokes system). *A smooth set of data* for the Navier-Stokes system up to time T is a triplet (u_0, f, T), where $0 < T < \infty$ is a time, the initial velocity vector field $u_0 : R^3 \to R^3$ and the forcing term $f : [0, T] \times R^3 \to R^3$ are assumed to be smooth on R^3 and $[0, T] \times R^3$ respectively (thus, u_0 is infinitely differentiable in space, and f is infinitely differentiable in space time), and u_0 is furthermore required to be divergence-free:

$$\nabla \cdot u_0 = 0.$$

If f = 0, we say that the data is *homogeneous.*
In the proofs of the main conjecture we will not consider any external force, thus the data will always be homogeneous. But we will state intermediate propositions with external forcing. Next we are defining simple diffentiability of the data by Sobolev spaces.
DEFINITION 3.2 We define the H^1 norm (or enstrophy norm) $H^1(u_0, f, T)$ of the data to be the quantity
$H^1(u_0, f, T) := \|u_0\|_{H^1_X(R^3)} + \|f\|_{L^{\infty}_t H^1_X(R^3)} < \infty$ and say that (u_0, f, T) *is H^1* if
$H^1(u_0, f, T) < \infty$.
DEFINITION 3.3 We say that a *smooth set of data* (u_0, f, T) is *Schwartz* if, for all integers α, m, $k \geq 0$, one has

$$\sup_{x\in R^3}(1+|x|)^k \left|\nabla_x^a u_0(x)\right| < \infty$$

and $$\sup_{(t,x)\in[0,T]\times R^3}(1+|x|)^k \left|\nabla_x^a \partial_t^m f(x)\right| < \infty$$

Thus, for instance, the solution or initial data having Schwartz property implies having the H^1 property.

DEFINITION 3.4 A *smooth solution* to the Navier-Stokes system, or a *smooth solution* for short, is a quintuplet (u, p, u_0 , f, T), where (u_0, f, T) is a *smooth set of data*, and the velocity vector field u : [0, T] × R^3 → R^3 and pressure field p : [0, T]× R^3 → R are smooth functions on [0, T]× R^3 that obey the Navier-Stokes equation (eq. 1) but with external forcing term f,

$$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu\Delta u_i + f_i \quad (x\epsilon R^3 , t>=0 , n=3)$$

and also the incompressibility property (eq.2) on all of [0, T] × R^3 , but also the initial condition u(0, x) = u_0(x) for all x ∈ R^3

DEFINITION 3.5 Similarly, we say that (u, p, u_0, f, T*) is H^1* if the associated data (u_0, f, T*) is H^1* , and in addition one has

$$\|u\|_{L_t^\infty H_X^1([0,T]\times R^3)} + \|u\|_{L_t^2 H_X^2([0,T]\times R^3)} < \infty$$

We say that the solution is *incomplete in [0,T),* if it is defined only in [0,t] for every t<T.

We use here the notation of *mixed norms* (as e.g. in TAO, T 2013). That is if $\|u\|_{H_x^k(\Omega)}$ is the classical Sobolev norm ,of smooth function of a spatial domain Ω, $u:\Omega\to R$, I is a time interval and $\|u\|_{L_t^p(I)}$ is the classical L^p -norm, then the mixed norm is defined by

$$\|u\|_{L_t^p H_x^k(I\times\Omega)} := (\int_I \|u(t)\|_{H_x^k(\Omega)}^p dt)^{1/p}$$

and

$$\|u\|_{L_t^\infty H_x^k(I\times\Omega)} := ess\sup_{t\in I}\|u(t)\|_{H_x^k(\Omega)}$$

Similar instead of the Sobolev norm for other norms of function spaces.

We also denote by $C_x^k(\Omega)$, for any natural number $k \geq 0$, the space of all k times continuously differentiable functions $u:\Omega\to R$, with finite the next norm

$$\|u\|_{C_x^k(\Omega)} := \sum_{j=0}^{k}\|\nabla^j u\|_{L_x^\infty(\Omega)}$$

We use also the next notation for *hybrid norms*. Given two normed spaces X, Y on the same domain (in either space or time), we endow their intersection $X \cap Y$ with the norm

$$\|u\|_{X\cap Y} := \|u\|_X + \|u\|_Y .$$

In particular in the we will use the next notation for intersection functions spaces, and their hybrid norms.

$X^k(I\times\Omega) := L_t^\infty H_x^k(I\times\Omega)\cap L_x^2 H_x^{k+1}(I\times\Omega)$.

We also use the *big O notation*, in the standard way, that is X=O(Y) means $X \leq CY$ for some constant C. If the constant C depends on a parameter s, we denote it by C_s and we write $X=O_s(Y)$.

We denote the difference of two sets A, B by A\B. And we denote Euclidean balls by $B(a,r) := \{x \in R^3 : |x-a| \leq r\}$, where |x| is the Euclidean norm.

With the above terminology the target Clay millennium conjecture in this paper can be restated as the next proposition

The 4th Clay millennium problem (Conjecture A)

(**Global regularity for homogeneous Schwartz data**). *Let (*u_0*, 0, T) be a homogeneous Schwartz set of data. Then there exists a smooth finite energy solution (u, p,* u_0*, 0, T) with the indicated data (notice it is for any T>0, thus global in time).*

4. Some known or directly derivable, useful results that will be used.

In this paragraph I state ,some known theorems and results, that are to be used in this paper, so that the reader is not searching them in the literature and can have a direct, at a glance, image of what holds and what is proved.

A review of this paragraph is as follows:

Propositions 4.1, 4.2 are mainly about the uniqueness and existence locally of smooth solutions of the Navier-Stokes and Euler equations with smooth Schwartz initial data. Proposition 4.3 are necessary or sufficient or necessary and sufficient conditions of regularity (global in time smoothness) for the Euler equations without viscosity. Equations 8-15 are forms of the energy conservation and finiteness of the energy loss in viscosity or energy dissipation. Equations 16-18 relate quantities for the conditions of regularity. Proposition 4.4 is the equivalence of smooth Schwartz initial data with smooth compact support initial data for the formulation of the 4th Clay millennium problem. Propositions 4.5-4.9 are necessary and sufficient conditions for regularity, either for the Euler or Navier-Stokes equations, while Propositions 4.10 is a necessary and sufficient condition of regularity for only the Navier-Stokes with non-zero viscidity.

In the next I want to use, the basic local existence and uniqueness of smooth solutions to the Navier-Stokes (and Euler) equations , that is usually referred also as the well posedness, as it corresponds to the existence and uniqueness of the physical reality causality of the flow. The theory of well-posedness for smooth solutions is summarized in an adequate form for this paper by the Theorem 5.4 in TAO, T. 2013.

I give first the definition of **mild solution** as in TAO, T. 2013 page 9. Mild solutions must satisfy a condition on the pressure given by the velocities. Solutions of smooth initial Schwartz data are always mild, but the concept of mild solutions is a generalization to apply for non-fast decaying in space initial data , as the Schwartz data, but for which data we may want also to have local existence and uniqueness of solutions.

DEFINITION 4.1

We define a *H^1 mild solution* (u, p, u_0, f, T) to be fields u, f :[0, T] × R^3 → R^3, p : :[0, T] × R^3 → R, u_0 : R^3 → R^3, with 0 < T < ∞ , obeying the regularity hypotheses

$u_0 \in H_x^1(R^3)$

$f \in L_t^\infty H_x^1([0,T]\times R^3)$

$u \in L_t^\infty H_x^1 \cap L_t^2 H_x^2([0,T]\times R^3)$

with the pressure p being given by (Poisson)

$$p = -\Delta^{-1}\partial_i\partial_j(u_i u_j) + \Delta^{-1}\nabla\cdot f \qquad \text{(eq. 7)}$$

(Here the summation conventions is used , to not write the Greek big Sigma).
which obey the incompressibility conditions (eq. 2), (eq. 3) and satisfy the integral form of the Navier-Stokes equations

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-t')\Delta}(-(u\cdot\nabla)u - \nabla p + f)(t')dt'$$

with initial conditions u(x,0)=u^0(x) **.**

We notice that the definition holds also for the in viscid flows, satisfying the Euler equations. The viscosity coefficient here has been normalized to ν**=1.**

In reviewing the local well-posedness theory of H^1 mild solutions, the next can be said. The content of the theorem 5.4 in TAO, T. 2013 (that I also state here for the convenience of the reader and from which derive our PROPOSITION 4.2) is largely standard (and in many cases it has been improved by more powerful current well-posedness theory). I mention here for example the relevant research by PRODI G 1959 and SERRIN,J 1963, The local existence theory follows from the work of KATO, T. PONCE, G. 1988 , the regularity of mild solutions follows from the work of LADYZHENSKAYA, O. A. 1967 . There are now a number of advanced local well-posedness results at regularity, especially that of KOCH, H., TATARU, D.2001. There are many other papers and authors that have proved the local existence and uniqueness of smooth solutions with different methods. As it is referred in C. L. FEFFERMAN 2006 I refer too the reader to the MAJDA A.J-BERTOZZI A. L. 2002 page 104 Theorem 3.4,

I state here for the convenience of the reader the summarizing theorem 5.4 as in TAO T. 2013. I omit the part (v) of Lipchitz stability of the solutions from the statement of the theorem. I use the standard O() notation here, x=O(y) meaning x<=cy for some absolute constant c. If the constant c depends on a parameter k, we set it as index of O_k().

It is important to remark here that the existence and uniqueness results locally in time (well-posedness) , hold also not only for the case of viscous flows following the

Navier-Stokes equations, but also for the case of inviscid flows under the Euler equations. There are many other papers and authors that have proved the local existence and uniqueness of smooth solutions both for the Navier-Stokes and the Euler equation with the same methodology , where the value of the viscosity coefficient v=0, can as well be included. I refer e.g. the reader to the MAJDA A.J-BERTOZZI A. L. 2002 page 104 Theorem 3.4 , paragraph 3.2.3, and paragraph 4.1 page 138.

PROPOSITION 4.1 *(Local well-posedness in H^1). Let (u_0, f, T) be H^1 data.*

(i) (Strong solution) If (u, p, u_0, f, T) is an H^1 mild solution, then

$$u \in C_t^0 H_x^1([0,T]\times R^3)$$

(ii) (Local existence and regularity) If

$$(\|u_0\|_{H_X^1(R^3)} + \|f\|_{L_t^1 H_X^1(R^3)})^4 T < c$$

for a sufficiently small absolute constant c > 0, then there exists a H^1 mild solution (u, p, u_0, f, T) with the indicated data, with

$$\|u\|_{X^k([0,T]\times R^3)} = O(\|u_0\|_{H_X^1(R^3)} + \|f\|_{L_t^1 H_X^1(R^3)})$$

and more generally

$$\|u\|_{X^k([0,T]\times R^3)} = O_k(\|u_0\|_{H_X^k(R^3)}, \|f\|_{L_t^1 H_X^1(R^3)}, 1)$$

for each k>=1 . In particular, one has local existence whenever T is sufficiently small, depending on the norm H^1(u_0, f, T).

(iii) (Uniqueness) There is at most one H^1 mild solution (u, p, u_0, f, T) with the indicated data.

(iv) ***(Regularity****) If (u, p, u_0, f, T) is a H^1 mild solution, and (u_0, f, T) is (smooth) Schwartz data, then u and p is smooth solution; in fact, one has*

$$\partial_t^j u, \partial_t^j p \in L_t^\infty H^k([0,T]\times R^3)$$ *for all j, K >=0.*

For the proof of the above theorem, the reader is referred to the TAO, T. 2013 theorem 5.4, but also to the papers and books , of the above mentioned other authors**.**

Next I state the local existence and uniqueness of smooth solutions of the Navier-Stokes (and Euler) equations with smooth Schwartz initial conditions , that I will use in this paper , explicitly as a PROPOSITION 4.2 here.

PROPOSITION 4.2 **Local existence and uniqueness of smooth solutions or smooth well posedness**. *Let* ***$u_0(x)$*** *,* ***$p_0(x)$*** *be smooth and Schwartz initial data at t=0* *of the Navier-Stokes (or Euler) equations, then there is a finite time interval [0,T] (in general depending on the above initial conditions) so that there is a unique smooth local in time solution of the Navier-Stokes (or Euler) equations*

$\mathbf{u(x)}$, $\mathbf{p(x)}$ $\in C^{\infty}(R^3 \times [0,T])$

Proof: We simply apply the PROPOSITION 4.1 above and in particular , from the part (ii) and the assumption in the PROPOSITION 4.2, that the initial data are smooth Schwartz , we get the local existence of H^1 mild solution (u, p, u_0, 0, T). From the part (iv) we get that it is also a smooth solution. From the part (iii), we get that it is unique.
As an alternative we may apply the theorems in MAJDA A.J-BERTOZZI A. L. 2002 page 104 Theorem 3.4 , paragraph 3.2.3, and paragraph 4.1 page 138, and getthe local in time solution, then derive from the part (iv) of the PROPOSITION 4.1 above, that they are also in the classical sense smooth. QED.

Remark 4.1 We remark here that the property of smooth Schwartz initial data, is not in general conserved in later times than t=0, of the smooth solution in the Navier-Stokes equations, because it is a very strong fast decaying property at spatially infinity. But for lower rank derivatives of the velocities (and vorticity) we have the **(global and) local energy estimate** , and **(global and) local enstrophy estimate** theorems that reduce the decaying of the solutions at later times than t=0, at spatially infinite to the decaying of the initial data at spatially infinite. See e.g. TAO, T. 2013, Theorem 8.2 (Remark 8.7) and Theorem 10.1 (Remark 10.6).
Furthermore in the same paper of formal formulation of the Clay millennium conjecture , L. FEFFERMAN 2006 (see page 3rd line 6 from above), it is stated that the 3D global regularity of such smooth solutions is controlled by the **bounded accumulation in finite time intervals** of the vorticity (Beale-Kato-Majda). I state this also explicitly for the convenience of the reader, for smooth solutions of the Navier-Stokes equations with smooth Schwartz initial conditions, as the PROPOSITION 4.6
When we say here bounded accumulation e.g. of the deformations D, **on finite internals**, we mean in the sense e.g. of the proposition 5.1 page 171 in the book MAJDA A.J-BERTOZZI A. L. 2002 , which is a definition designed to control the existence or not of finite blowup times. In other words for any finite time interval
[0, T], there is a constant M such that

$$\int_0^t |D|_{L^\infty}(s)ds <= M$$

I state here for the convenience of the reader, a well known proposition of equivalent necessary and sufficient conditions of existence globally in time of solutions of the Euler equations, as inviscid smooth flows. It is the proposition 5.1 in MAJDA A.J-BERTOZZI A. L. 2002 page 171.

The *stretching* is defined by

$S(x,t) =: D\xi \cdot \xi$ if $\xi \neq 0$ and $S(x,t) =: 0$ if $\xi = 0$ where $\xi =: \frac{\omega}{|\omega|}$, ω being the vortcity.

PROPOSITION 4.3 *Equivalent Physical Conditions for Potential Singular Solutions of the Euler equations . The following conditions are equivalent for smooth Schwartz initial data:*

(1) The time interval, $[0, T^*)$ *with* $T^* < \infty$ *is a maximal interval of smooth* H^s *existence of solutions for the 3D Euler equations.*

(2) The vorticity ω accumulates so rapidly in time that

$$\int_0^t |\omega|_{L^\infty}(s)ds \to +\infty \text{ as t tends to T*}$$

(3) The deformation matrix D accumulates so rapidly in time that

$$\int_0^t |D|_{L^\infty}(s)ds \to +\infty \text{ as t tends to T*}$$

*(4) The stretching factor S(**x**, t) accumulates so rapidly in time that*

$$\int_0^t [\max_{x \in R^3} S(x,s)]ds \to +\infty \text{ as t tends to T*}$$

The next theorem establishes the equivalence of smooth connected compact support initial data with the smooth Schwartz initial data, for the homogeneous version of the 4th Clay Millennium problem. It can be stated either for local in time smooth solutions or global in time smooth solutions. The advantage assuming connected compact support smooth initial data, is obvious, as this is preserved in time by smooth functions and also integrations are easier when done on compact connected sets.

PROPOSITION 4.4. **(3D global smooth compact support non-homogeneous regularity implies 3D global smooth Schwartz homogeneous regularity)** *If it holds that the incompressible viscous (following the Navier-Stokes equations) 3 dimensional local in time [0,T] , finite energy, flow-solutions with smooth compact support (connected with smooth boundary) initial data of velocities and pressures (thus finite initial energy) and smooth compact support (the same connected support with smooth boundary) external forcing for all times t>0, exist also globally in time t>0 (are globally regular) then it also holds that the incompressible viscous (following the Navier-Stokes equations) 3 dimensional local in time [0,T] , finite energy, flow-solutions with smooth Schwartz initial data of velocities and pressures (thus finite initial energy) , exist also globally in time for all t>0 (are regular globally in time).*

(for a proof see KYRITSIS, K. 2017, PROPOSITION 6.4)

Remark 4.2 Finite initial energy and energy conservation equations:
When we want to prove that the smoothness in the local in time solutions of the Euler or Navier-Stokes equations is conserved, and that they can be extended indefinitely in time, we usually apply a "reduction ad absurdum" argument: Let the maximum finite time T* and interval [0,T*) so that the local solution can be extended smooth in it.. Then the time T* will be a blow-up time, and if we manage to extend smoothly the solutions on [0,T*]. Then there is no finite Blow-up time T* and the solutions holds in [0,+∞). Below are listed necessary and sufficient conditions for this extension to be possible. Obviously not smoothness assumption can be made for the time T*, as this is what must be proved. But we still can assume that at T* the energy conservation and momentum conservation will hold even for a singularity at T*, as these are universal laws of nature, and the integrals that calculate them, do not require smooth functions but only integrable functions, that may have points of discontinuity.
A very well known form of the energy conservation equation and accumulative energy dissipation is the next:

$$\frac{1}{2}\int_{R^3}\|u(x,T)\|^2 dx+\int_0^T\int_{R^3}\|\nabla u(x,t)\|^2 dxdt=\frac{1}{2}\int_{R^3}\|u(x,0\|^2 dx \qquad \text{(eq. 8)}$$

where

$$E(0)=\frac{1}{2}\int_{R^3}\|u(x,0)\|^2 dx \qquad \text{(eq. 9)}$$

is the initial finite energy

$$E(T)=\frac{1}{2}\int_{R^3}\|u(x,T)\|^2 dx \qquad \text{(eq. 10)}$$

is the final finite energy

and $\Delta E=\int_0^T\int_{R^3}\|\nabla u(x,t)\|^2 dxdt$ (eq. 11)

is the accumulative finite energy dissipation from time 0 to time T , because of viscosity in to internal heat of the fluid. For the Euler equations it is zero. Obviously

ΔE<=E(0)>=E(T) (eq. 12)

The rate of energy dissipation is given by

$$\frac{dE}{dt}(t)=-v\int_{R^3}\|\nabla u\|^2 dx<0 \qquad \text{(eq. 13)}$$

(v, is the viscosity coefficient. See e.g. MAJDA, A.J-BERTOZZI, A. L. 2002 Proposition 1.13, equation (1.80) pp. 28)

At this point we may discuss, that for the smooth local in time solutions of the Euler equations, in other words for flows without viscosity, it is paradoxical from the physical point of view to assume, that the total accumulative in time energy dissipation is zero while the time or space-point density of the energy dissipation (the former is the $\|\nabla u(x,t)\|^2_{L_\infty}$), is not zero! It is indeed from the physical meaningful point of view unnatural, as we cannot assume that there is a loss of energy from to viscosity at a point and a gain from “anti-viscosity” at another point making the total zero. Neither to assume that the time and point density of energy dissipation is non-zero or even infinite at a space point, at a time, or in general at a set of time and space points of measure zero and zero at all other points, which would still make the total accumulative energy dissipation zero. **The reason is of course that the absence of viscosity, occurs at every point and every time, and not only in an accumulative energy level.** If a physical researcher does not accept such inviscid solutions of the Euler equation as having physical meaning, then for all other solutions that have physical meaning and the $\|\nabla u(x,t)\|^2_{L_\infty}$ is zero (and come so from appropriate initial data), we may apply the PROPOSITION 4.7 below and **deduce directly, that the local in time smooth solutions of the Euler equations, with smooth Schwartz initial data, and finite initial energy, and zero time and space point energy dissipation density due to viscosity, are also regular (global in time smooth).** For such regular inviscid solutions, we may see from the inequality in (eq. 15) below, that the total L2-norm of the vorticity is not increasing by time. We capture this remark in PROPOSITION 4.11 below.

Remark 4.3 The next are 3 very useful inequalities for the unique local in time [0,T], smooth solutions u of the Euler and Navier-Stokes equations with smooth Schwartz initial data and finite initial energy (they hold for more general conditions on initial data, but we will not use that):

By $\|.\|_m$ we denote the Sobolev norm of order m. So if m=0 itis essentially the L_2-norm. By $\|.\|_{L\infty}$ we denote the supremum norm, u is the velocity, ω is the vorticity, and cm, c are constants.

1) $\|u(x,T)\|_m \le \|u(x,0)\|_m \exp(\int_0^T c_m \|\nabla(u(x,t)\|_{L_\infty} dt)$ (eq. 14)

(see e.g. MAJDA, A.J-BERTOZZI, A. L. 2002 , proof of Theorem 3.6 pp117, equation (3.79))

2) $\|\omega(x,t)\|_0 \le \|\omega(x,0)\|_0 \exp(c\int_0^T \|\nabla u(x,t)\|_{L_\infty} dt)$ (eq. 15)

(see e.g. MAJDA, A.J-BERTOZZI, A. L. 2002 , proof of Theorem 3.6 pp117, equation (3.80))

3) $\|\nabla u(x,t)\|_{L_\infty} \le \|\nabla u(x,0)\|_0 \exp(\int_0^t \|\omega(x,s)\|_{L_\infty} ds)$ (eq. 16)

(see e.g. MAJDA, A.J-BERTOZZI, A. L. 2002 , proof of Theorem 3.6 pp118, last equation of the proof)

The next are a list of well know necessary and sufficient conditions , for regularity (global in time existence and smoothness) of the solutions of Euler and Navier-Stokes equations, under the standard assumption in the 4th Clay Millennium problem of smooth Schwartz initial data, that after theorem Proposition 4.4 above can be formulated equivalently with smooth compact connected support data. We denote by T* be the maximum Blow-up time (if it exists) that the local solution u(x,t) is smooth in [0,T*).

1) PROPOSITION 4.5 **(Necessary and sufficient condition for regularity)**
The local solution u(x,t) , t in [0,T) of the Euler or Navier-Stokes equations, with smooth Schwartz initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if the* ***Sobolev norm ||u(x,t)||$_m$*** *, m>=3/2+2 , remains bounded , by the same bound in all of [0,T*), then , there is no maximal Blow-up time T*, and the solution exists smooth in [0,+∞)*

Remark 4.4 See e.g. . MAJDA, A.J-BERTOZZI, A. L. 2002 , pp 115, line 10 from below)

2) PROPOSITION 4.6 **(Necessary and sufficient condition for regularity.** Beale-Kato-Majda**)**
The local solution u(x,t) , t in [0,T) of the Euler or Navier-Stokes equations, with smooth compact connected support initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if for the finite time interval [0,T*], there exist a bound M>0, so that the* ***vorticity has bounded by M, accumulation*** *in [0,T*]:*

$$\int_0^{T^*} \|\omega(x,t)\|_{L_\infty} dt \le M \qquad (eq17)$$

Then there is no maximal Blow-up time T, and the solution exists smooth in [0,+∞)*
Remark 4.5 See e.g. . MAJDA, A.J-BERTOZZI, A. L. 2002 , pp 115, Theorem 3.6. Also page 171 theorem 5.1 for the case of inviscid flows. . See also LEMARIE-RIEUSSET P.G. 2002 . Conversely if regularity holds, then in any interval from the smoothness in a compact connected set, the vorticity is supremum bounded. The above theorems in the book MAJDA A.J-BERTOZZI A. L. 2002 guarantee that the above conditions extent the local in time solution to global in time , that is to solutions (u, p, u_0, f, T) which is H^1 mild solution, **for any T**. Then applying the part (iv) of the PROPOSITION 4.1 above, we get that this solutions is also smooth in the classical sense, for all T>0, thus globally in time smooth.

3) PROPOSITION 4.7 **(Necessary and sufficient condition for regularity)**
The local solution u(x,t) , t in [0,T) of the Euler or Navier-Stokes equations, with smooth compact connected support initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if for the finite time interval [0,T*], there exist a bound M>0, so that the* ***vorticity is bounded by M, supremum norm*** *L∞ in [0,T*]:*

$$\|\omega(x,t)\|_{L_\infty} \le M \text{ for all t in [0,T*)} \qquad (eq. 18)$$

Then there is no maximal Blow-up time T, and the solution exists smooth in [0,+∞)*

Remark 4.6 Obviously if $\|\omega(x,t)\|_{L_\infty} \leq M$, then also the integral exists and is bounded: $\int_0^{T^*} \|\omega(x,t)\|_{L_\infty} dt \leq M_1$ and the previous proposition 4.6 applies. Conversely if regularity holds, then in any interval from smoothness in a compact connected set, the vorticity is supremum bounded.

4) PROPOSITION 4.8 **(Necessary and sufficient condition for regularity)**
The local solution u(x,t) , t in [0,T) of the Euler or Navier-Stokes equations, with smooth compact connected support initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if for the finite time interval [0,T*], there exist a bound M>0, so that the space accelerations are bounded by M, in the supremum norm L∞ in [0,T*]:*

$\|\nabla u(x,t)\|_{L_\infty} \leq M$ *for all t in [0,T*)* *(eq. 19)*

Then there is no maximal Blow-up time T, and the solution exists smooth in [0,+∞)*

Remark 4.7 Direct from the inequality (eq.14) and the application of the proposition 4.5. Conversely if regularity holds, then in any finite time interval from smoothness, the accelerations are supremum bounded.

5) PROPOSITION 4.9 **(**FEFFERMAN C. L. 2006. **Necessary and sufficient condition for regularity)**
The local solution u(x,t) , t in [0,T) of the Navier-Stokes equations with non-zero viscosity, and with smooth compact connected support initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if*
the velocities ||u(x,t)|| do not get unbounded as t->T.*
Then there is no maximal Blow-up time T, and the solution exists smooth in [0,+∞).*

Remark 4.8.This is mentioned in the Official formulation of the 4^{th} Clay Millennium problem FEFFERMAN C. L. 2006 pp.2 , line 1 from below: quote "...For the Navier-Stokes equations (v>0) , if there is a solution with a finite blowup time T, then the velocities $u_i(x,t)$, 1<=i<=3 become unbounded near the blowup time." The converse-negation of this is that if the velocities remain bounded near the T*, then there is no Blowup at T* and the solution is regular or global in time smooth. Conversely of course , if regularity holds, then in any finite time interval, because of the smoothness, the velocities, in a compact set are supremum bounded.
I did not find a dedicated such theorem in the books or papers that I studied, but since prof. C.L Fefferman , who wrote the official formulation of the 4^{th} Clay Millennium problem, was careful to specify that is in the case of non-zero viscosity v>0, and not of the Euler equations as the other conditions, I assume that he is aware of a proof of it.

6) PROPOSITION 4.10. **(Necessary condition for regularity)**
Let us assume that the local solution u(x,t) , t in [0,T) of the Navier-Stokes equations with non-zero viscosity, and with smooth compact connected support initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), in other words that are regular, then the trajectories-paths length l(a,t) does not get unbounded as t->T*.*
Proof: Let us assume that the solutions is regular. Then also for all finite time intervals [0,T] , the velocities and the accelerations are bounded in the L_∞ , supremum norm, and this holds along all trajectory-paths too. Then also the length of the trajectories , as they are given by the formula

$$l(a_0,T)=\int_0^T \|u(x(a_0,t)\|dt \qquad \text{(eq. 20)}$$

are also bounded and finite (see e.g. APOSTOL T. 1974 , theorem 6.6 p128 and theorem 6.17 p 135). Thus if at a trajectory the lengths becomes unbounded as t goes to T*, then there is a blow-up. QED.

7) PROPOSITION 4.11.**(Physical meaningful inviscid solutions of the Euler equations are regular)**
Let us consider the local solution u(x,t) , t in [0,T) of the Euler equations with zero viscosity, and with smooth compact connected support initial data. If we conside,r because of zero-viscosity at every space point and at every time, as physical meaningful solutions those that also the time and space points energy dissipation density, due to viscosity, is zero or* $\|\nabla u(x,t)\|^2_{L_\infty}=0$ *, then , they can be extended smooth to all times [0,+∞), in other words they are regular.*
Proof: Direct from the PROPOSITION 4.8. QED.

Remark 4.9.
Similar results about the local smooth solutions, hold also for the non-homogeneous case with external forcing which is nevertheless space-time smooth of bounded accumulation in finite time intervals. Thus an alternative formulation to see that the velocities and their gradient , or in other words up to their 1st derivatives and the external forcing also up to the 1st derivatives , control the global in time existence is the next proposition. See TAO. T. 2013 Corollary 5.8

PROPOSITION 4.12 (**Maximum Cauchy development**)
Let (u_0, f, T) be H^1 data. Then at least one of the following two statements hold:
1) There exists a mild H^1 solution (u, p, u_0, f, T) in [0,T] ,with the given data.
2)There exists a blowup time $0 < T^* < T$ and an incomplete mild H^1 solution (u, p, u_0, f, T^*) up to time T^* in [0, T^*), defined as complete on every [0,t], $t<T^*$ which blows up in the enstrophy H^1 norm in the sense that

$$\lim_{t\to T^*,t<T^*}\|u(x,t)\|_{H^1_x(R^3)}=+\infty$$

Remark 4.10 The term "almost smooth" is defined in TAO, T. 2013, before Conjecture 1.13. The only thing that almost smooth solutions lack when compared to smooth solutions is a limited amount of time differentiability at the starting time t = 0; The term *normalized pressure*, refers to the symmetry of the Euler and Navier-Stokes equations to substitute the pressure, with another that differs at, a constant in space but variable in time measureable function. In particular normalized pressure is one that satisfies the (eq. 7) except for a measurable at a, constant in space but variable in time measureable function. It is proved in TAO, T. 2013, at Lemma 4.1, that the pressure is normalizable (exists a normalized pressure) in almost smooth finite energy solutions, for almost all times. The viscosity coefficient in these theorems of the above TAO paper has been normalized to $\nu=1$**.**

5. Conservation of the particles as a local structure of fluids in the context of continuous fluid mechanics. Proof of the regularity for fluids with conservation of particles as a local structure, and the hypotheses of the official formulation of the 4th Clay millennium problem, for the Euler and Navier-Stokes equations.

Remark 5.1 (**Physical interpretation of the definition 5.1**) The smoothness of the particle-trajectory mapping (or displacement transformation of the points), the smoothness of the velocity field and vorticity field, is a condition that involves statements in the orders of micro scales of the fluid, larger, equal and also by far smaller that the size of material molecules, atoms and particles, from which it consists. This is something that we tend to forget in continuous mechanics, because continuous mechanics was formulated before the discovery of the existence of material atoms. On the other-hand it is traditional to involve the atoms and particles of the fluid, mainly in mathematical models of statistical mechanics. Nevertheless we may formulate properties of material fluids in the context of continuous fluid mechanics, that reflect approximately properties and behavior in the flow of the material atoms. This is in particular the DEFINITION 5.1. **For every atom or material particle of a material fluid, we may assume around it a ball of fixed radius, called *particle range* depending on the size of the atom or particle, that covers the particle and a little bit of the electromagnetic, gravitational or quantum vacuum field around it, that their velocities and space-time accelerations are affected by the motion of the molecule or particle.** E.g. for the case water, we are speaking here for molecules of H_2O, that are estimated to have a diameter of 2.75 angstroms or 2r= 2.75*10^(-10) meters, we may define as water molecule **particle range** the balls $B(r_0)$ of radius r_0=3*10^(-10) meters around the water molecule. As the fluid flows, especially in our case here of incompressible fluids, the shape and size of the molecules do not change much, neither there are significant differences of the velocities and space-time accelerations of parts of the molecule. Bounds δ_u δ_ω of such differences remain constant as the fluid flows. **We may call this effect as the *principle of conservation of particles* as a local structure.** This principle must be posed in equal setting as the energy conservation and incompressibility together with the Navier-Stokes or Euler equations. Of course if the fluid is say of solar plasma matter, such a description would not apply.

Nevertheless then incompressibility is hardly a property of it. But if we are talking about incompressible fluids that the molecule is conserved as well as the atoms and do not change atomic number (as e.g. in fusion or fission) then this principle is physically valid. The principle of conservation of particles as a local structure, blocks the self-similarity effects of concentrating the energy and turbulence in very small areas and creating thus a Blow-up. It is the missing invariant in the discussion of many researchers about superctitical, critical and subcritical invariants in scale transformations of the solutions.
The next DEFINITION 5.1 formulates precisely mathematically this principle for the case of incompressible fluids.

DEFINITION 5.1. (**<u>Conservation of particles as local structure in a fluid</u>**)
Let a smooth solution of the Euler or Navier-Stokes equations for incompressible fluids, that exists in the time interval [0,T). We may assume initial data on all of R^3 or only on a connected compact support V_0 . For simplicity let us concentrate only on the latter simpler case. Let us denote by F the displacement transformation of the flow Let us also denote by g the partial derivatives of 1^{st} order in space and time , that is $\left|\partial_x^a \partial_t^b u\ (x)\right|$, $|\alpha|=1$, $|b|<=1$,and call then space-time accelerations . We say that there is ***<u>conservation of the particles</u>*** ***<u>in the interval [0,T) in</u>*** *a derivatives homogenous setting, as a local structure of the solution if and only if:*
There is a small radius r, and small constants δ_x , δ_u , δ_ω , >0 so that for all t in [0,T) there is a finite cover C_t (in the case of initial data on R^3 ' it is infinite cover, but finite on any compact subset) of V_t , from balls B(r) of radius r, called ***<u>ranges of the particles</u>*** *, such that:*

4) *For an x_1 and x_2 in a ball B(r) of V_s , s in [0,T), $||F(x_1)-F(x_2)||<=r+\delta_x$ for all t>=s in [0,T).*
5) *For an x_1 and x_2 in a ball B(r) of V_s ,s in [0,T), $||u(F(x_1))-u(F(x_2))||<=\delta_u$ for all t >=s in [0,T).*
6) *For an x_1 and x_2 in a ball B(r) of V_s , s in [0,T), $||g(F(x_1))-g(F(x_2))||<=\delta_\omega$ for all t >=s in [0,T).*

   If we state the same conditions 1) 2) 3) for all times t in [0,+∞) , then we say that we have the ***<u>strong version</u>*** *of the conservation of particles as local structure.*

PROPOSITION 5.1 (**Velocities on trajectories in finite time intervals with finite total variation, and bounded in the supremun norm uniformly in time.)**
Let u_t : V(t) -> R^3 be smooth local in time in [0,T) ,velocity fields solutions of the Navier-Stokes or Euler equations, with compact connected support V(0) initial data, finite initial energy E(0) and conservation of particles in [0,T*) as a local structure . The [0,T*) is the maximal interval that the solutions are smooth. Then for t in [0,T*) and x in V(t), the velocities are uniformly in time bounded in the supremum norm by a bound M independent of time t.*

$$\|u(x,t)\|_{L_\infty} = \sup_{x \in V(t)} \|u(x,t)\| \leq M \quad \textit{for all t in } [0,T^*).$$

Therefore the velocities on the trajectory paths, in finite time intervals are of bounded variation and the trajectories in finite time interval, have finite length.

1st Proof (Only for the Navier-Stokes Equations): Let us assume, that the velocities are unbounded in the supremum norm, as t converges to T*. Then there is a sequence of times t_n with t_n converging to time T* , and sequence of corresponding points x_n (t_n), for which the norms of the velocities ||u(x_n (t_n), t_n)|| converge to infinite.

$$\lim_{n\to+\infty} \| u(x(x_n,t_n),t_n) \| = +\infty . \qquad \text{(eq.21)}$$

From the hypothesis of the conservation of particles as a local structure of the smooth solution in [0,T*), for every t_n There is a finite cover C_{tn} of particle ranges, of V_{tn} so that x_n (t_n) belongs to one such ball or particle-range B_n(r) and for any other point y(t_n) of B_n(r), it holds that ||u(x_n (t_n), t_n)-u(y(t_n),t_n)||<= δ_u . Therefore

$$\|u(x_n(t_n), t_n)\| - \delta_u <= \|u(y(t_n),t_n)\| <= \|u(x_n(t_n), t_n)\| + \delta_u \qquad \text{(eq.22)}$$

for all times t_n in [0,T*) .

By integrating spatially on the ball B_n(r), and taking the limit as n->+∞ we deduce that

$$\lim_{n\to+\infty} \int_{B_n} \|u\| dx = +\infty$$

But this also means as we realize easily, that also

$$\lim_{n\to+\infty} \int_{B_n} \|u\|^2 dx = +\infty \qquad \text{(eq. 23)}$$

Which nevertheless means that the total kinetic energy of this small, but finite and of constant radius, ball, converges to infinite, as t_n converges to T*. This is impossible by the finiteness of the initial energy, and the conservation of energy. Therefore the velocities are bounded uniformly ,in the supremum norm, in the time interval [0,T*).

Therefore the velocities on the trajectory paths, are also bounded in the supremum norm , uniformly in the time interval [0,T*). But this means by PROPOSITION 4.9 that the local smooth solution is regular , and globally in time smooth, which from PROPOSITION 4.8 means that the Jacobian of the 1st order derivatives of the velocities are also bounded in the supremum norm uniformly in time bounded in [0,T*). Which in its turn gives that the velocities are of bounded variation on the trajectory paths (see e.g. APOSTOL T. 1974 , theorem 6.6 p128 and theorem 6.17 p 135) and that the trajectories in have also finite length in [0,T*), because the trajectory length is given by the formula $l(a_0,T)=\int_0^T \|u(x(a_0,t)\|dt$. QED.

2nd Proof (Both for the Euler and Navier-Stokes equations): Instead of utilizing the condition 2) of the definition 5.1, we may utilize the condition 3). And we start assuming that the Jacobian of the velocities is unbounded in the supremum norm (instead of the velocities), as time goes to the Blow-up time T*. Similarly we conclude that the energy dissipation density at a time on balls that are particle-ranges goes to infinite, giving the same for the total accumulative in time energy dissipation (see (eq. 11), which again is impossible from the finiteness of the initial energy and energy conservation. Then by PROPOSITION 4.8 we conclude that the solution is regular, and thus also that the velocities are bounded in the supremum norm, in all finite time intervals. Again we deduce in the same way, that the total variation of the velocities is finite in finite time intervals and so are the lengths of the trajectories too. QED.

PROPOSITION 5.2 (**Global regularity as in the 4th Clay Millennium problem).**

Let the Navier-Stokes or Euler equations with smooth compact connected initial data , finite initial energy and conservation of particles as local structure. Then the unique local in time solutions are also regular (are smooth globally in time).

Proof: We apply the PROPSOITION 5.1 above and the necessary and sufficient condition for regularity in PROPSOITION 4.9 (which is only for the Navier-Stokes equations). Furthermore we apply the part of the 2d proof of the PROPOSITION 5.1 , that concludes regularity from PROPSOITION 4.8 which holds for both the Euler and Navier-Stokes equations QED.

6. Bounds of measures of the turbulence from length of the trajectory paths, and the total variation of the velocities, space acceleration and vorticity. The concept of homogeneous smoothness.

Remark 6.1 In the next we define **a measure of the turbulence** of the trajectories, of the velocities, of space-time accelerations and of the vorticity, through the **total variation** of the component functions in finite time intervals. This is in the context of deterministic fluid dynamics and not stochastic fluid dynamics. We remark that in the case of a blowup the measures of turbulence below will become infinite.

DEFINITION 6.1 **(The variation measure of turbulence)**

Let smooth local in time in [0,T] solutions of the Euler or Navier-Stokes equations. The total length L(P) of a trajectory path P, in the time interval [0,T] is defined as **the variation measure of turbulence of the displacements** on the trajectory P, in [0,T].The total variation TV(||u||) of the norm of the velocity ||u|| on the trajectory P in [0,T] is defined as **the variation measure of turbulence of the velocity** on the trajectory P in [0,T]. The total variation TV(g) of the space-accelerations g (as in DEFINITION 5.1) on the trajectory P in [0,T] is defined as **the variation measure of turbulence of the space-time accelerations** on the trajectory P in [0,T]. The total variation TV(||ω||) of the norm of the vorticity ||ω|| on the trajectory P in [0,T] is defined as **the variation measure of turbulence of the vorticity** on the trajectory P in [0,T].

PROPOSITION 6.1 **Conservation in time of the boundedness of the maximum turbulence, that depend only on the initial data and time lapsed.**
Let the Euler or Navier-Stokes equations with smooth compact connected initial data finite initial energy and conservation of the particles as a local structure. Then for all times t, there are bounds $M_1 (t)$, $M_2 (t)$, $M_3(t)$, so that the maximum turbulence of the trajectory paths , of the velocities and of the space accelerations are bounded respectively by the above universal bounds, that depend only on the initial data and the time lapsed.

Proof: From the PROPOSITIONS 5.1, 5.2 we deduce that the local in time smooth solutions are smooth for all times as they are regular. Then in any time interval [0,T], the solutions are smooth, and thus from the PROPOSITION 4.8, the space acceleration g, are bounded in [0,T], thus also as smooth functions their total variation TV(g) is finite, and bounded. (see e.g. APOSTOL T. 1974 , theorem 6.6 p128 and theorem 6.17 p 135).From the PROPOSITION 4.7, the vorticity is smooth and bounded in [0,T], thus also as smooth bounded functions its total variation TV(||ω||) is finite, and bounded on the trajectories. From the PROPOSITION 4.9, the velocity is smooth and bounded in [0,T], thus also as smooth bounded functions its total variation TV(||u||) is finite, and bounded on the trajectories. From the PROPOSITION 4.10, the motion on trajectories is smooth and bounded in [0,T], thus also as smooth bounded functions its total variation which is the length of the trajectory path L(P) is finite, and bounded in [0,T].In the previous theorems the bounds that we may denote them here by $M_1 (t)$, $M_2 (t)$, $M_3(t)$, respectively as in the statement of the current theorem, depend on the initial data, and the time interval [0,T]. QED.

Remark 6.2. (Homogeneity of smoothness relative to a property P.) There are many researchers that they consider that the local smooth solutions of the Euler or Navier-Stokes equations with smooth Schwartz initial data and finite initial energy,(even without the hypothesis of conservation of particles as a local structure) are general smooth functions. But it is not so! They are special smooth functions with the remarkable property that there are some critical properties P_i that if such a property holds in the time interval [0,T) for the coordinate partial space-derivatives of 0, 1, or 2 order , then this property holds also for the other two orders of derivatives. In other words if it holds for the 2 order then it holds for the orders 0, 1 in [0,T) . If it holds for the order 1, then it holds for the orders 0, 2 in [0,T]. If it holds for the order 0, them it holds also for the orders 1,2 in [0,T]. This pattern e.g. can be observed for the property P_1 of uniform boundedness in the supremum norm, in the interval [0,T*) in the PROPOSITIONS 4.5-4.10 . But one might to try to prove it also for a second property P_2 which is the **finitness of the**

total variation of the coordinates of the partial derivatives, or even other properties P3 like **local in time Lipchitz conditions**. This creates a strong bond or coherence among the derivatives and might be called ***homogeneous smoothness relative to a property P.*** We may also notice that the formulation of the conservation of particles as local structure is in such a way, that as a property, it shows the same pattern of homogeneity of smoothness relative to the property of uniform in time bounds P_4 ,1), 2), 3) in the DEFINITION 5.1. It seem to me though that even this strong type of smoothness is not enough to derive the regularity, unless the homogeneity of smoothness is relative to the property P_4, in other words the conservation of particles as a local structure.

7. Epilogue

I believe that the main reasons of the failure so far in proving of the 3D global regularity of incompressible flows, with reasonably smooth initial conditions like smooth Schwartz initial data, and finite initial energy, is hidden in the difference of the physical reality ontology that is closer to the ontology of statistical mechanics models and the ontology of the mathematical models of continuous fluid dynamics. Although energy and momentum conservation and finiteness of the initial energy are easy to formulate in both types of models, the conservation of particles as type and size is traditionally formulated only in the context of statistical mechanics. By succeeding in formulating approximately in the context of the ontology of continuous fluid mechanics the conservation of particles during the flow, as local structure, we result in being able to prove the regularity in the case of 3 dimensions which is what most mathematicians were hoping that it holds.

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