ALTERNATIVE OPTION PRICING METHODS, BASED ON THE CONCEPT OF INSURANCE

Abstract

In this paper, are given, alternative methods to the Black-Scholes method of option pricing, yielding the latter as special case. The alternative methods are similar to the methods of insurance policies pricing in actuarial mathematics. The choice of the model that represents the changes of the price of the underlying exchange market is left open. Numerical examples are given and the proposed method is compared to the traditional Black-Scholes method. It is made a discussion about the resulting advantages, and about possible applications in weather derivatives.

Περίληψη

Σε αυτή την εργασία, δίνουμε εναλλακτικές μεθόδους σε σχέση με το μοντέλο Black-Scholes θεωρητικής τιμολογησης δικαιωμάτων στα παράγωγα, οι οποίες δίνουν το τελευταίο ως ειδική περίπτωση. Οι εναλλακτικές αυτές μέθοδοι βασίζονται στα ασφαλιστικά μαθηματικά. Η επιλογή του μοντέλου που ακολουθούν οι αλλαγές των τιμών της υποκείμενης χρηματιστηριακής αγοράς αφήνεται ανοικτή. Δίνονται αφθονικά παραδείγματα μέσα από προσομοίωση και η προτεινόμενη μέθοδος συγκρίνεται με το παραδοσιακό μοντέλο Black-Scholes. Τέλος, γίνεται συζήτηση για τα πλεονεκτήματα που προκύπτουν και για πιθανές εφαρμογές στα μετεωρολογικά (καιρικά) παράγωγα.

JEL Classification: G22, G13

key words: Financial Options, Black-Scholes option pricing, insurance, actuarial mathematics, weather derivatives.
1. Introduction.

It is often said that derivatives are used either for speculation or hedging. In particular, it seems that from the derivatives, options were originally designed as insurance contracts for the case of loss or damage in positions on other assets, indexes, commodities or securities, but especially of futures. We say especially for futures, because positions in futures can easily lead to bankruptcy, due to the existence of margin and leverage, while positions in securities, literally, do not. By buying contracts of put options we can insure long positions in futures while by buying contracts of call options we can insure short positions on futures. This paper requires a familiarity with derivatives, either as practical experience or education on them. For the definitions of put and call options, the reader may refer to Hull (2000) (chapter 1, pp 1-5) or Wilmot (1999). We do require this familiarity so as not to have to define in this paper terms like margin, leverage, market maker etc. Alternative models to the Black-Scholes model have been published by Cox and Ross (1976), by Duan (1995), by Hull and White (1987) etc., all of which are based on different ideas, than the present paper’s idea. The authors, introduce in this paper, a new model for option fair pricing, which is based on principles of actuarial mathematics and insurance, and which includes the Black-Scholes as a special case. The main advantage of the present suggested model, is that the final option’s fair price, does depend and includes, the trend (drift) of the underlying asset. In the Black-Scholes model, the option’s fair price is independent from the trend of the underlying asset, and the only way for the market-makers to include it in the computation of a fair price, is to use different volatilities of the underlying asset (volatility smiles etc), for different strike-prices. This nevertheless is a contradiction of the definition of volatility for the underlying and a fairly complicated and irrational technique. In the present approach, the market maker, can make use, of one only volatility, for the underlying asset, plus a trend of it, that can be positive or negative, and accounts for his expected, or forecasted trend of the market for the underlying asset, till the expiration. The practice of the market- makers as far as their forecasting of the trend of the underlying, through adjustments of the parameter of volatility, of the Black-Scholes formulae, is not based on any standard statistical forecasting technique, but rather on arbitrary intuitive choices. In the present approach the market maker could make use of a statistical measurement of the trend of the underlying, as he can do for the volatility too. This is pretty easier in weather derivatives where the underlying is based e.g. on the temperature of a place, as seasonal temperatures are easier to forecast. But obviously, the widespread tendency is to put an arbitrary expected trend, at each time, according to their speculations. From this point of view the present suggested approach for the option fair price, does not include necessarily, a better forecasting for the underlying, as usually, there is not a standard forecasting technique for this. Of course in the approach suggested in this paper, we could use more appropriate forecasting techniques for the underlying asset, than those used by the Black-
Scholes model, or we could use exactly the same forecasting. The present option fair pricing does suggest nevertheless a better system of choices for the market maker or the investor, based on two parameters rather than one, the trend and the volatility of the underlying. When the trend of the underlying is put equal to the risk-free rate, then the present formulae for the option's fair price coincide with those of Black-Scholes model.

The present suggested option fair pricing, based on principles of insurance, is not included of course in any treatise or publication (as far as we know) of actuarial mathematics, as it is standard that in the applications of actuarial mathematics is not included the Financial Derivatives, and options, that is a topic of finance rather than insurance. Some ideas similar to the ideas in this paper have been published in option pricing of weather derivatives, where most of the researchers and practitioners agree that the Black-Scholes model is not the appropriate.


The model that is used by the market makers of most markets for options fair pricing is the Black-Scholes model. An outline of the characteristics of this model is given below.

Before we proceed we have to outline the model for the movements of underlying assets that is assumed in the Black-Scholes Model as well, and is based on the Brownian motion or Wiener process.

0. Underlying assets follow a geometric Brownian motion. The geometric Brownian motion is like a random geometric progression. of, say a random interest compound. The exact definition is the following:

1. If $S$ is the spot price of an asset, it is assumed that $S$ follows an (Ito) stochastic process (see Oksendal (1995))

$$dS = \mu S dt + \sigma S dz$$

(1)

where $dz$ is a Wiener process or Brownian motion. In other words it has the next two properties

a) The increment or change $\Delta z$ during time $\Delta t$ is $\varepsilon \cdot \sqrt{\Delta t}$, where $\varepsilon$ is a standardized normal random variable

b) The increments $\Delta z$ for two different intervals of time $\Delta t$ are independent, as random variables.

Equivalently, we could postulate that the spot price of the asset follows a geometric Brownian or Wiener process, in other words, the logarithm $\ln S$ of the spot price follows a Wiener or Brownian process, generally with drift $\mu$ [see also Karlin and Taylor (1975), p. 357]. $\mu$ is called the drift of the process and $\sigma$ its coefficient of volatility.
The drift parameter $\mu$ and the coefficient of volatility $\sigma$ in discrete time and a sufficient fine grid or resolution of time step $\delta t$, can be estimated by the formulae:

$$\mu = \frac{1}{N\delta t} \sum_{i=1}^{N} R_i$$  \hspace{1cm} (2),

where $R_i = \frac{S_{i+1} - S_i}{S_i}$  \hspace{1cm} (3) and

$$\sigma = \sqrt{\frac{1}{(N-1)\delta t} \sum_{i=1}^{N} (R_i - \bar{R})^2}$$  \hspace{1cm} (4)

where $\bar{R} = \mu \delta t$  \hspace{1cm} (5)

Obviously if the time unit is the pixel $\delta t$, then $\mu$ and $\sigma$ are the average and standard deviation of the rates of return per time step which is also the time unit.

For the option fair pricing model of Black-Scholes, the following axioms are assumed:

1) The stock or underlying price follows a geometric Wiener process with volatility $\sigma$ and drift $\mu$.

2) The short selling of the underlying asset is permitted

3) There are no transaction costs or taxes, and the invested value of the underlying can be of any continuous size (the values of contracts are not necessarily integers)

4) There are no dividends during the life cycle of the option.

5) There are no riskless arbitrage opportunities

6) Trading of the underlying is continuous

7) The risk-free rate is constant during the life cycle of the option.

As a result of the above axioms, it can be proved that an option price $f$ has to satisfy the next Partial Differential equation known also as Black-Scholes-Merton equation.

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \left(\frac{1}{2}\right)\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$  \hspace{1cm} (6)

The solution of this equation with boundary final condition $f(S_t, T) = \max(S_t - X, 0)$ when $t=T$, which is the payoff for a standard European call option at expiry, and at time $t=0$ is:
\[ c = S_0 N(d1) - X e^{-rT} N(d2) \] (7)

where

\[ d_2 = \frac{\ln(S_0 / X) + (r - \sigma^2 / 2)T}{\sigma \sqrt{T}} \] (8)
\[ d_1 = \frac{\ln(S_0 / X) - (r - \sigma^2 / 2)T}{\sigma \sqrt{T}} \] (9)

and \( N(x) \) is the distribution function of a normal random variable.

Various modifications to this model have been presented. For example, Huang and Chen (2002) propose the use of a model having a stochastic volatility parameter. Lehar et al. (2002) propose the GARCH model and compare it to the stochastic volatility model. A common point of reference for these works is the use of a volatility parameter, on which certain assumptions are imposed.

3. Introduction to the insurance model of Option Pricing.

In the B-S option fair pricing model, the drift of the underlying is different in general from the risk-free continuous time rate, but after all the axioms of the model is does not enter eventually in the final formula of fair price.

An alternative approach would be to abandon the erroneous axioms of the B-S model and to allow the drift of the underlying asset to enter the formula. The fact that options are actually insurance contracts of loss in long or short positions, gives an obvious way to derive their fair premium price, in a way exactly the same with insurance contracts in actuarial mathematics. The principle of pricing insurance contracts in actuarial mathematics is based on the following simple equation:

The average value of the outflow from the insurance company to the customer due to the occurrence of the events being insured, discounted in present value should be equal to the sum of inflows in the company from the customer, discounted in present value.

This principle of course requires a model to estimate the probabilities of the insured events, and thus derive the average value of the outflows from the company to the customer. In the case of options this model is the model of the price changes of the underlying asset.

We may restate the previous general principle for the case of options with which we derive option fair prices:
THE GENERAL INSURANCE PRINCIPLE OF OPTION FAIR PRICING.

The options fair price \( P \) at a present time moment \( t \) is the discount with the risk-free rate \( q \) at this moment \( t \), of the average paid value at expiration \( T \), given that the value of the price random variable, \( S_t \), of the underlying asset at expiration is calculated by an assumed model \( M \) and by the spot price \( S \), at the present time \( t \).

We notice here that the above concept of fair pricing corresponds to buying the contract of the option at time \( t \) and exercising it at expiration, without intermediate trading. This way, no assumptions about zero transaction costs or continuous delta-hedging trading or infinite continuous divisibility of invested size, are necessary.

As we shall see in the next paragraph, the Black-Scholes option fair pricing falls too under this general principle, if we assume that the model \( M \) is a geometric Brownian motion with a drift equal to the continuous-time, risk-free rate. Although in the assumptions of the B-S model the geometric Brownian motion of the underlying asset does not necessarily have a drift equal to the risk-free rate, the final fair premium when included in the above principle corresponds to a choice of geometric Brownian motion with drift equal to the risk-free rate.

3. Black-Scholes model: a special case of the insurance option pricing model.

To prove that the Black-Scholes option fair price formula is a special case of the above general insurance principle of option fair pricing, it is required to state the following lemma (Hull (2000), p.268).

**Lemma:** If \( V \) is lognormal distributed with average value \( m \) and the standard deviation of \( \ln V \) is \( s \), then

\[
E[\max(V - X, 0)] = mN(d_1) - XN(d_2)
\]

where

\[
d_1 = \frac{\ln(\frac{m}{X}) + \frac{s^2}{2}}{s}
\]

\[
d_2 = \frac{\ln(\frac{m}{X}) - \frac{s^2}{2}}{s}
\]

\( E[A] \) denotes the average value of the random variable \( A \) and \( N(x) \) is the distribution function of a normal variable \( x \).

Using this result we may derive the general form of the option fair price, and the exact formula in the case we assume (as is also assumed in the B-S model) that the underlying follows a geometric Brownian motion (lognormal distribution).
If we interpret the average value as the average value of the price of one unit of the underlying (one item of the security, if the underlying is a stock) at expiration, and the standard deviation as the standard deviation of the price of the underlying at expiration, then the average value of payoff of one contract of a call option at expiration of exercise price \( X \) (and assuming that one contract insures one item of the underlying) is

\[
P = E\left[ \max(V - X, 0) \right]
\]

(12)

The general principle of pricing requires of course to have this average payoff value discounted at present values, if the pricing is not at expiration. So if the current time is \( t \), and expiration is time \( T \), while the risk-free (continuous time) rate by which we discount is \( \rho \), then the present fair price of the option is,

\[
P = e^{-\rho(T-t)} E\left[ \max(V - X, 0) \right]
\]

(13)

This is the general formula of the option fair price, which is model-free in the sense that any model may be assumed for the changes of the prices of the underlying.

If we assume that the price of the underlying at expiration is also lognormal distributed with average value \( m \) and standard deviation \( s \), then we may apply the above lemma to transform the general formula into:

\[
P = e^{-\rho(T-t)} (mN(d_1) - XN(d_2))
\]

(14)

where

\[
d_1 = \frac{\ln\left(\frac{m}{X}\right) + s^2}{s^2} \quad d_2 = \frac{\ln\left(\frac{m}{X}\right) - s^2}{s^2}
\]

(15)

This is again a general formula of the option fair price, which is also model-free in the sense that, we may assume any model for the changes of the prices of the underlying, provided it is lognormal distributed at expiration, with average value \( m \) and standard deviation \( s \).

Let us now choose a particular model for the changes of the prices of the underlying, which is the same as the one assumed by the Black-Scholes model, namely a geometric Brownian motion of drift \( \mu \) and volatility \( \sigma \). Then the average value of the price of the underlying at expiration \( T \) is

\[
m = S e^{(\mu - \frac{1}{2}\sigma^2)(T-t)}
\]

(16)

where \( t \) is the present time and \( S \) is the present value of the underlying. This yields the following formula for the fair option's premium:

\[
P = Se^{-\rho(T-t)} (e^{(\mu - \frac{1}{2}\sigma^2)(T-t)} N(d_1) - XN(d_2))
\]

(17)
where
\[ d_1 = \frac{\ln(S/e^{rT}) + (\rho + s^2/2)}{s\sqrt{T-t}} \]  
\[ d_2 = \frac{\ln(S/e^{rT}) - (\rho - s^2/2)}{s\sqrt{T-t}} \]

(18)  
(19)

The reader should notice that both the drift \( r \) and the risk-free rate \( \rho \) enter the formula; in general, these two figures are different.

If we now assume that the drift of the underlying \( r \) is equal to the risk-free rate \( \rho \), then the above formula reduces to the familiar formula of option fair price (for call options) of the Black-Scholes Model (Wilmot (1999), p 97):

\[ P = N(d_1) - Xe^{-\rho(T-t)}N(d_2) \]

(20)

where
\[ d_1 = \frac{\ln(S/X) + (\rho + s^2/2)}{s\sqrt{T-t}} \]  
\[ d_2 = \frac{\ln(S/X) - (\rho - s^2/2)}{s\sqrt{T-t}} \]

(21)  
(22)

Thus we have proved:

The insurance model of option fair pricing has as special case the Black-Scholes model, when in the insurance model we assume that the drift of the underlying is equal to the risk-free rate.


In this section, some numerical comparisons between the Black-Scholes model and the insurance model of option fair pricing are presented. In particular we notice that the underlying (chart 1) has negative descending trend. And the options fair price as computed by the present insurance model (lower series 2 of chart 2) is indeed of lower price as it should, compared to that the option’s fair price computed by the Black-Scholes model (Middle series 1 of chart 2). According to the assumptions of the models both option prices are fair, but usually the market-makers when expecting a descending trend of the underlying, as it is here, the alter the volatility of the underlying (which here is put 40% as it is measured), to a lower value, thus achieving the effect, of a lower fair price for the option. With the suggested model, it is not necessary to put a lower volatility, but rather a negative trend for the underlying. So the advantage is rather not of a better forecasting (as in both models exactly the same forecasting for the underlying can be used), but a better way to vary the option fair price according to the speculations of the market maker, or investor, in a logically consistent way, and given that it is the standard practice of the market
makers to do so, but only in a artificial and logically inconvenient way through the parameter of the volatility. Of course in the insurance option fair pricing we could make use of more appropriate techniques of forecasting of the underlying, than this in the Black-Scholes model, or we could use exactly the same forecasting.

We take real data of the underlying asset, that in this case is the index FTSE/ase20 (high capitalization) of the Athens Stock Exchange Market for the time interval from January 3rd, 2002 to January 22nd, 2002. Using this data, we estimate the daily option fair price by Black-Scholes model (for a call option of a fixed exercise price) and, parallel to it, the option fair price by the insurance model, with a drift for FTSE estimated on an initial previous time interval. These series of data are presented in Table 1. The calculations below have been carried out using a software application written by the first of the authors, who used Visual Basic for Applications to develop an Option Simulator in the environment of Microsoft Office Excel.

<table>
<thead>
<tr>
<th>Date</th>
<th>Ftse-20</th>
<th>Call Option Premium by Black-Scholes</th>
<th>Call Option Premium drift -4% by Insurance Model</th>
<th>Call Option Premium drift 8% by Insurance Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>03-01-02</td>
<td>1441,71</td>
<td>70,22</td>
<td>67,40</td>
<td>71,57</td>
</tr>
<tr>
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<td>72,44</td>
<td>69,74</td>
<td>73,73</td>
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<tr>
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<td>66,29</td>
<td>63,89</td>
<td>67,45</td>
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<tr>
<td>08-01-02</td>
<td>1423,27</td>
<td>52,17</td>
<td>50,22</td>
<td>53,11</td>
</tr>
<tr>
<td>09-01-02</td>
<td>1407,41</td>
<td>40,56</td>
<td>39,02</td>
<td>41,31</td>
</tr>
<tr>
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<td>37,89</td>
<td>36,53</td>
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<tr>
<td>11-01-02</td>
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<td>29,56</td>
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<tr>
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<td>14,22</td>
<td>15,09</td>
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<td>11,78</td>
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<td>16-01-02</td>
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<td>22-01-02</td>
<td>1392,14</td>
<td></td>
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</tr>
</tbody>
</table>

Table 1: Values for the FTSE/ase20 (high capitalization) index of the Athens Stock Exchange Market from 3/1/02 to 22/1/02 and estimates for the daily option fair price resulting from the different models.

The values for the FTSE-20 is plotted as a time series in Chart 1, while the premiums for the three models are plotted in Chart 2.
Chart 1: The diagram for the FTSE-20 index.

Chart 2: Theoretical prices for Call options, using different models
The reader may notice the sensitivity of the fair price of the insurance model on the trend of the stock market. If the trend is ascending, the insurance model yields a higher price for the call option, thus accounting for the fact that the option is likely to be exercised yielding a higher profit to the investor. If the trend is descending, then the fair price is lower, since it is likely that the stock market will go lower, thus making the call option unattractive to the investor.

In the table above we took at random a time interval of 14 days of the index FTSE-20 (High Capitalization) of the Athens Stock Exchange market for the time interval 3-01 to 21-01 of 2002 (14 Days). We made the assumption that a call option expires at the end of this time interval and we estimated the corresponding fair prices of such an option. A volatility of 40% for the underlying and a risk-free rate at 4% are assumed. The middle line in chart 2 (data in column 2 of the table) is the fair price calculated by the Black-Scholes Model. The lower line (column 3 of the table) is the fair price calculated by the insurance model with assumed drift for the underlying equal to -4% (downwards trend). The upper line is the fair price by the insurance model with assumed drift for the underlying equal to 8% (upward trend). Although the differences for this range of time-to-expire are small, the additional parameter of drift of the underlying allows for flexibility in defining the fair price, as a function of the markets trend. In fact, the drift can be automatically estimated for a given past time interval (e.g. of the same length as at the estimation of the historic volatility) by least squares best fit. There are standard formulae for the determination of the drift at the geometric Brownian Motion (see formula 2). It is important to notice that, in trying to adjust the options price to the trend of the market, the market makers traditionally vary the assumed volatility, which results in changing in an undesired way both the call and the put options prices; this is clearly undesirable and leads to adopting the use of different volatilities for different exercise prices. This is essentially a violation of the definition of the volatility in the Black-Scholes model as referring to the underlying asset, and thus being independent from the exercise price of the options. The proposed model has been used for investment purposes for a year (2001), in the fair option pricing in the Athens Derivatives exchange, by the first of the authors.

Another interesting direction of applications is in the option pricing of weather derivatives. There is a growing interest in weather derivatives, due to the increased frequency of extreme weather phenomena, and the need to insure business related to the price of raw materials like crude oil or agricultural commodities. Most of the researchers and practitioners in weather derivatives agree that the Black-Scholes model is not an appropriate model for the case of weather derivatives. The main reason seems to be that e.g. an underlying based on the temperature of a city, is an almost perfect random periodic stochastic process with period of one year and not a geometric
Brownian motion. Various approaches have been proposed (see [DAVIS M], [RICHARDS T.J. etc], [KAMINSKY, V]) some completely different and some quite close to the proposed option pricing method, in this paper. The idea of an option pricing method, quite flexible so as to accommodate for any model of the stochastic process of the underlying instrument, makes our approach quite general and capable to give as special case, many of the published alternative option pricing methods for weather derivatives.

5. Conclusion.

We have seen how we can formulate an alternative option fair pricing model based on the basic equation of pricing of premiums in insurance contracts with the following characteristics

1) It is left open which is the model of price movements of the underlying, while of course we can as well assume a geometric Brownian motion, or even an other non-Markovian model like ARMA models of time series that may include periodic models of oscillations of the average value or the autocovariances, of the underlying instrument, appropriate for weather derivatives. (see e.g. [DUAN J-C (1995)])

2) As in this method of option pricing we may put different models for the underlying instrument, the fair option price can be sensitive to whether the spot market is descending, ascending or neither. For example, if the model for the underlying is the geometric Brownian motion, it is sensitive to its drift parameter. If it is a periodic stochastic process, it is sensitive to the period, phase, amplitude etc. This permits easier variation of the option fair price by easily measurable parameters of the spot market, and therefore more flexible option fair pricing.

3) The above approach gives an alternative formulation and derivation of the Black-Scholes option fair pricing formula that completely avoids the ironic, ambiguous and controversial Black-Scholes assumptions about risk less arbitrage opportunities, continuous delta hedging trading, zero transaction costs and infinite continuous divisibility of invested size. To derive exactly the Black-Scholes option fair pricing formula we only require the assumption that in the average the underlying has a drift, equal to the risk-free rate. To account for other drifts, and with the same natural assumptions we should resort to the insurance option fair pricing.

4) The present approach is very useful in calculating new option pricing methods for weather derivatives, and it may give as special case some of the already published alternative methods for the case of weather derivatives.
Bibliography


